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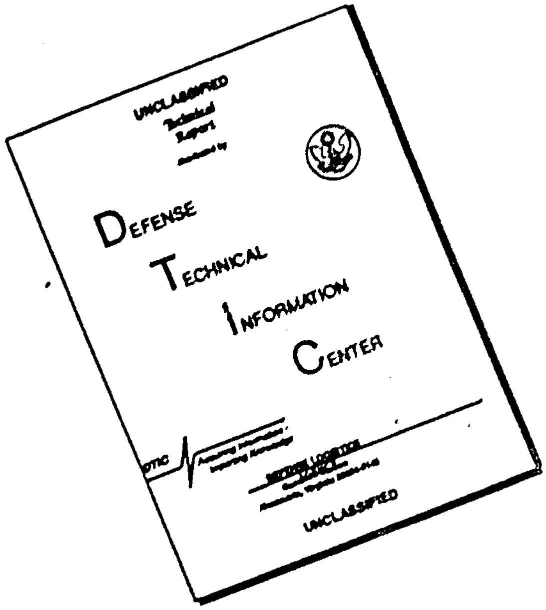

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TECHNICAL NOTE

GEOMETRICAL AND PHYSICAL INTERPRETATION OF THE WEYL CONFORMAL CURVATURE TENSOR

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ABSTRACT

The Weyl (conformal curvature) tensor of space-time is interpreted geometrically in terms of the behaviour of congruences of null geodesics. The corresponding physical interpretation provides, in principle, a means for the measurement of physical components of the Weyl tensor with light rays alone, without the use of clocks or rigid rods.

GEOMETRICAL AND PHYSICAL INTERPRETATION OF
THE WEYL CONFORMAL CURVATURE TENSOR

by

F.A.E. Pirani and A. Schild

1. Introduction: Conformal-invariant methods

This paper outlines, without proofs, a geometrical and physical interpretation of the Weyl tensor in the space-time of general relativity. Proofs and further details will be published elsewhere. The interpretation embodies a method whereby the physical components of the Weyl tensor could, in principle, be measured by observations of light-rays alone, without the use of clocks or rigid rods. The possibility of such measurements may be inferred from the fact that null geodesics, which represent light rays, are invariant under conformal transformations of Riemannian space-time, while proper time along time-like lines is not invariant (nor are time-like geodesics invariant). In vacuum, the same measurements will yield corresponding components of the Riemann curvature tensor, since the Riemann and Weyl tensors coincide wherever Einstein's vacuum field equations hold.

In a previous paper (Pirani 1956), a physical interpretation was given to the Riemann tensor, and a method of measuring its components by observations of test particles was described. The

whole discussion there was explicitly metrical, and required the use of clocks (or rigid rods) as well as light rays. It was understood that a particular Riemannian space-time was given in advance, and the arguments depended on the measurement of proper times.

In the present paper, on the other hand, the arguments are entirely <u>conformal-invariant</u>; both geometrical and physical interpretations refer not to a particular Riemannian space-time, but to a whole class of space-times which may be obtained from one another by conformal transformations of the metric. Such a class of Riemannian space-time is called a <u>conformal space-time</u>; thus a conformal space-time $C_{(4)}$ is a (sufficiently) differentiable manifold endowed at each of its points P with a real infinitesimal <u>null cone</u>[1]

$$g_{ab}dx^a dx^b = 0. \tag{1}$$

Physically, the null cone is the history of a wave front of light collapsing to and emitted from the event P. The quadratic form (1) must have hyperbolic normal signature in order that the proper distinctions between past and future may be preserved.

The null cone (1) determines the metric $e^{2\sigma} g_{ab}$ in any of the Riemannian space-times of the conformal class up to a <u>gauge factor</u>, $e^{2\sigma}$ which is an arbitrary function of position. A

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1. Latin indices a,b,c,... range and sum over 1,2,3,4.

particular Riemannian space-time may be selected by assigning the gauge. It is often easier to carry out proofs of theorems in a particular gauge, exhibiting their conformal-invariance afterwards, than to devise a strictly conformal-invariant proof.

It is evident that the ratio of the magnitudes of two vectors or of two simple bivectors at the same point, and the angle between two directions at the same point (especially, the orthogonality of two directions at the same point) are well-defined in a conformal space-time. It is in fact easy to show how these quantities may be determined by experiments with light signals alone.

In § 2 we give conformal-invariant definitions of null geodesics and of preferred parameters on them. In § 3 we give a conformal-invariant definition of infinitesimal shear, and state our main result, which connects the second parameter-derivative of the shear with the conformal curvature tensor. The infinitesimal shear was introduced by Sachs (1961 a,b) in his analysis of null geodesic congruences, and many of the ideas employed here were developed originally, in metrical form, by him.

2. Null geodesics and preferred parameters

A null hypersurface is defined, conformal-invariantly, as a hypersurface which is tangent at each of its points to the infinitesimal null cone at that point. Equivalently, a null hypersurface contains at each of its points exactly one null direction. A null geodesic is a null curve which lies entirely in a null

hypersurface (this manifestly conformal-invariant definition reduces to the usual one as soon as a gauge is assigned). Physically, a null geodesic is the world-line of a light ray, that is, the history of a light pulse.

We show now how to define a conformal-invariant preferred parameter u along any null geodesic of a given congruence. Let ℓ be the selected geodesic and Δx^a its infinitesimal tangent vector at any point P (Fig. 1). Let ℓ_1, ℓ_2, and ℓ_3 be three neighbouring null geodesics, ℓ_1 and ℓ_2 being chosen so that the connecting vectors $\delta_1 x^a$, $\delta_2 x^a$ from P to ℓ_1, ℓ_2 are orthogonal to Δx^a, and ℓ_3 being chosen so that the connecting vector $\delta_3 x^a$ from ℓ to ℓ_3 is not orthogonal to Δx^a. Then a parameter u may be defined along ℓ by the condition

$$(2) \qquad \frac{\text{area of parallelogram spanned by } \Delta x^a/\Delta u \text{ and } \delta_3 x^a}{\text{area of parallelogram spanned by } \delta_1 x^a \text{ and } \delta_2 x^a} = \text{constant along } \ell .$$

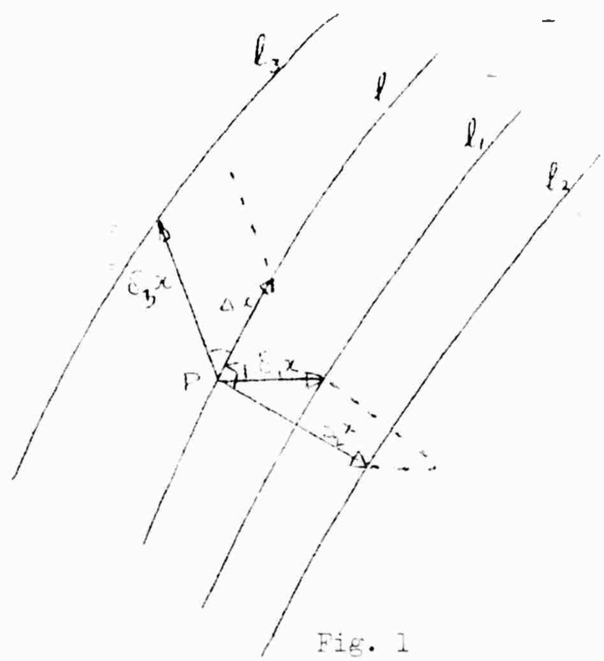

Fig. 1

It can be shown that this definition determines the parameter <u>u</u> up to a linear transformation with coefficients constant along ℓ, irrespective of the choice of the neighbouring geodesics ℓ_1, ℓ_2, ℓ_3 and of the choice of connecting vectors from P.

If the gauge is chosen so that the denominator of (2) is constant along ℓ (which corresponds to a zero magnification rate along ℓ in the terms of Sachs's analysis), then the parameter <u>u</u> may be identified with the usual preferred parameter along a null geodesic in the corresponding Riemannian space (cf Synge and Schild 1949, p.46).

3. Infinitesimal shear and its propagation

We now define in a conformal-invariant way the <u>infinitesimal shear</u> for a congruence of null geodesics (cf Sachs 1961a, p.). Let S_P and S_Q be infinitesimal 2-elements orthogonal to a null geodesic ℓ at neighbouring points P and Q of ℓ, and let C be an infinitesimal circle with centre P, lying in S_P (Fig. 2). Those null geodesics of the congruence which meet S_P in the circle C will meet S_Q in an ellipse E. The <u>infinitesimal shear</u> of the congruence from P to Q, $d\varepsilon$, is defined by the equation

$$(3) \quad \frac{\text{length of major axis of E}}{\text{length of minor axis of E}} = 1 + 2d\varepsilon.$$

It may be shown that both $d\varepsilon$ and the major axis e^a of the ellipse E are determined uniquely by the congruence of null geodesics and the points P and Q, and that $d\varepsilon$ is independent of the choice of the orthogonal 2-elements S_P and S_Q.

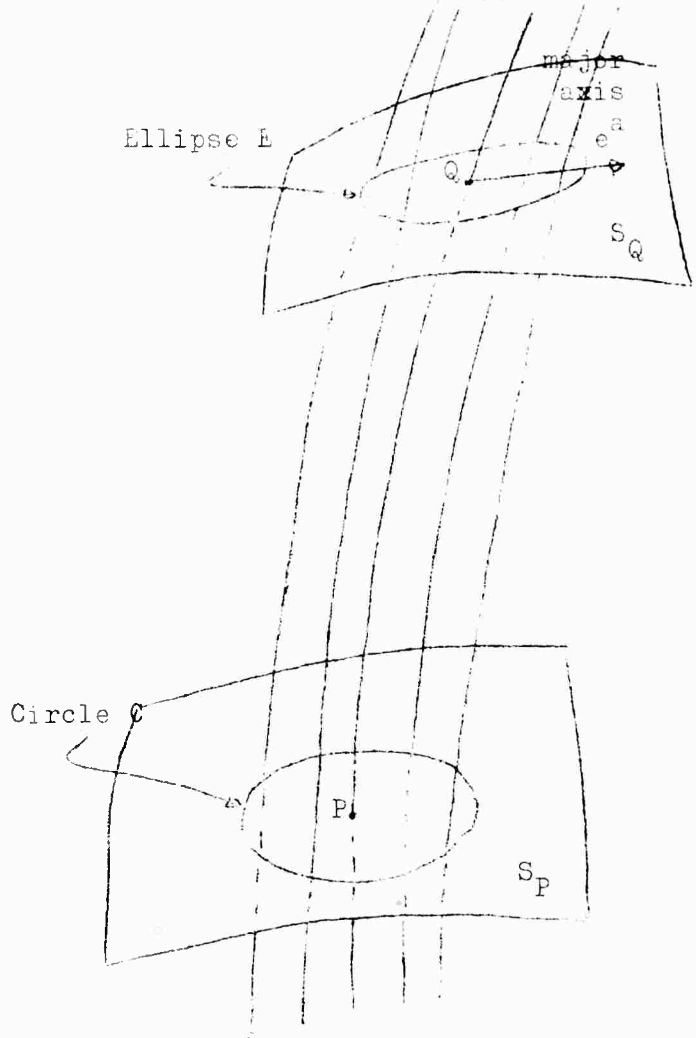

Fig. 2

The physical interpretation of this construction has been given, for the metrical case by Sachs (1961 a): Suppose that light passes normally through a small flat transparent circular disc and falls normally on a screen nearby. At a certain instant (P), the disc becomes momentarily opaque, and throws a shadow on the screen (Q). Refraction of the light by the gravitational field makes the shadow elliptical. The null geodesics represent the light rays; the circle C represents the periphery of the disc and the ellipse E the periphery of the shadow. Sachs has shown that the

shape, size and orientation of the shadow depend only on the choice of P and Q and not on the velocities of the disc or the screen. It is evident from the above construction that $d\varepsilon$ is conformal-invariant. It can be shown that the <u>orientation</u> of the shadow also is conformal-invariant; clearly, the size is not.

We can now state our main result, which relates the propagation of the shear along a null geodesic to the conformal tensor. It is that

$$(4) \qquad \frac{d^2\varepsilon}{du^2} = C^a{}_{bcd} \frac{dx^b}{du} \frac{dx^c}{du} P_a{}^b$$

Here $C^a{}_{bcd}$ is Weyl's conformal curvature tensor: in any Riemannian space of the conformal class,

$$C^a{}_{bcd} = R^a{}_{bcd} + g_{b[d}R^a{}_{c]} + R_{b[d}\delta^a{}_{c]} - {}^1/_3\, g_{b[d}\delta^a{}_{c]}R,$$

$R^a{}_{bcd}$ is the Riemann curvature tensor, defined for example by the commutation rule for covariant differentiation, $V_{b;cd} - V_{b;dc} = R^a{}_{bcd}V_a$, for any vector V_a, $R_{bc} = R^a{}_{bca}$ is the Ricci tensor, and $R = g^{bc}R_{bc}$. Also $\frac{dx^b}{du}$ is tangent vector to the geodesic, and $P_a{}^b$ is the projection operator in the direction of the major axis e^a of shear: $P_a{}^b e^a = e^b$, $P_a{}^b f^a = 0$ for every vector f^a orthogonal to e^a.

Equation (4) determines a physical component of the tensor $C^a{}_{bcd}$. Since $C^a{}_{bcd}$ is irreducible under transformations of the local orthogonal frame at any point (i.e. under Lorentz transformations), all the components of $C^a{}_{bcd}$ may be determined

by examining enough congruences of null geodesics, that is, by carrying out appropriate experiments with light rays.

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